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On Monoids, ... *John N. Van der Vries.*

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ON MONOIDS.

BY JOHN N. VAN DER VRIES.

(Continued.)

WE shall hereafter consider in each section all monoids that have the same lines of kinds III and IV, and shall subdivide the monoids in these sections according to the lines of kind I and the multiple lines of kind II which are found on them. When we write a line of kind I(1, 2), we mean the particular line of kind I which is a single line on the superior cone and a double line on the inferior cone.

Monoids having a double point, say the point (0, 0, 1, 0), *on a line* xy *of kind* III.

a). If the monoid has a double line of kind II, say the line xz, its equation will be of the form

$$x^2u_2 + xzv_2 + z^2w_2 + xzsv_1 + z^2sw_1 + x^2su_1 = 0.*$$

Every plane through the double line and one of the seven other lines of the monoid will in general cut a transversal out of the monoid, no two of the seven lines lying in general in one plane with the double line.

b). If the monoid has a second double line of kind II, say the line yz, the equation of the monoid will be of the form

$$x^2y^2 + xyzu_1 + z^2u_2 + z^2sv_1 + xyzs = 0.$$

The inferior cone breaks up into the plane z and a quadric cone. This monoid has two ordinary lines in addition to the three lines xy, xz, and yz. A plane through either double line and one of the two ordinary lines or the line of kind III cuts out a transversal from the monoid. The monoid thus has in general six transversals in addition to the five lines through the vertex.

*Hereafter u, v, w and t with subscripts will denote homogeneous functions of x and y of degrees equal to the subscripts, unless otherwise designated.

c). If the monoid has a simple line of kind I(1, 2), say the line xz, its equation will be of the form

$$xu_3 + zv_3 + z^2u_2 + x^2su_1 + xzsv_1 + z^2sw_1 = 0.$$

The monoid will have eight ordinary lines in addition; it will not in general have any transversals.

d). If the monoid has two single lines of kind I(1, 2), say xz and yz, the equation of the monoid will be of the form

$$xyu_2 + zu_3 + v_2z^2 + xyzs + u_1z^2s = 0.$$

The inferior cone breaks up into a plane and a quadric cone; the plane meets the monoid in four lines passing through the vertex. There are six ordinary lines on the monoid besides the three lines xy, xz, and yz. In general, there is no case in which three and only three of the nine lines on the monoid lie in one plane, and therefore there is in general no transversal on a monoid of this kind.

e). If the monoid has three simple lines of kind I(1, 2), the inferior cone will break up into three planes not passing through one and the same line. The intersections of these three planes are simple lines on the superior cone. The equation of this monoid will then be of the form

$$u_4 + u_3z + u_2z^2 + u_1v_1w_1s = 0;$$

where u_1, v_1 and w_1 are homogeneous functions of x, y and z of degree 1. One of the planes of the inferior cone meets the superior cone in four lines, of which two are lines of kind I, and other two the double line which is the line of kind III on the monoid. The other two planes meet the superior cone in four lines, of which two are lines of kind I and two are ordinary lines on the monoid. This accounts for all twelve lines on the monoid.

f). If the monoid has a simple line of kind I(1, 2), say yz, in addition to a double line, say xz, of kind II, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + z^2v_2 + xyzs + z^2sv_1 = 0.$$

The inferior cone breaks up into a plane and a quadric cone. A plane through the double line and the line of kind III cuts out of the monoid a transversal which passes through the double point on the line of kind III and intersects the double line at the point at which the scrolar tangent plane coincides with this cutting plane. A plane through the double line and the line of kind I cuts out of the monoid an ordinary line in addition. There are three other ordinary lines on the monoid, and no two of these will in general lie in the same plane with the double line. There are thus in general four transversals on this monoid.

g). If the monoid has two single lines of kind I(1, 2), in addition to the double line of kind II, the equation of the monoid will be of the form

$$u_4 + u_3z + u_2z^2 + u_1v_1w_1s = 0.$$

This monoid differs from case *e*) above in the fact that one of the double edges of the inferior cone is also a double edge of the superior cone. Two of the planes of the inferior cone meet the monoid in the double line, a line of kind I, and an ordinary line, whereas the third plane of the inferior cone meets the monoid in the line of kind III counting twice and two lines of kind I. A plane through the double line and either of the ordinary lines or the line of kind III cuts a transversal from the monoid.

The monoid can have no third line of kind I(1, 2).

h). If the monoid has a line of kind I(1, 2) in addition to two double lines of kind II, the equation will be similar to that in case *g*) above; two of the double edges of the inferior cone will however be double edges on the superior cone. One of the planes of the inferior cone meets the monoid in the two double lines, whereas the other two each cuts the monoid in a double line and a line of kind I or a line of kind III. This monoid has no transversals.

i). If the monoid has a simple line of kind I(1, 3), say xz, its equation will be of the form

$$xu_3 + zv_3 + z^2u_2 + x^3s + x^2zs + xz^2s = 0.$$

The inferior cone breaks up into three planes having the line xy in common. One of these planes meets the monoid in the line xy counting twice, the line xz and an ordinary line through the vertex, whereas the other two planes each cuts out of the monoid three ordinary lines in addition to the lines of kind III. This monoid can only have a transversal if three lines, one in each of the three planes of the inferior cone, lie in one plane.

j). If the monoid has a double line of kind I(2, 3), say xz, its equation will be of the form

$$x^2u_2 + xzv_2 + z^2w_2 + x^3s + x^2zs + xz^2s = 0.$$

The inferior cone breaks up into three planes which pass through one line, viz., xz. One of these planes meets the monoid in the lines xy and xz each counting twice, whereas the other two each cuts out the line xz twice and two ordinary lines. If two of these ordinary lines lie in the same plane with the line of kind III, there will be a transversal passing through the double point on the line of kind III.

k). If the monoid has a line xz of kind I(1, 3) and a line of kind I(1, 2), its equation will be of the form

$$xyu_2 + z^2v_2 + zv_3 + xz^2s = 0.$$

The inferior cone breaks up into a double plane and a single plane, the triple edge of the inferior cone being an ordinary edge on the superior cone, and the double edge of the superior cone being an ordinary edge on the inferior cone. The double plane meets the superior cone in the lines xz and yz, and in two additional lines, which are also lines of kind I(1, 2) on the monoid. The single plane meets the monoid in the line xy counting twice, the line xz, and an ordinary line $x(ay+bz)$. This monoid will not have any transversals.

Monoids having two double points, say the points (0, 0, 1, 0) *and* (0, 1, 0, 0), *on the lines* xy *and* xz *of kind* III.

a). If the monoid has a double line of kind II, say the line yz, its equation will be of the form

$$xyzu_1 + z^2u_2 + x^2y^2 + z^2u_1s + yzv_1s + xy^2s = 0.$$

The plane containing the lines of kind III cuts the monoid in addition in a conic which passes through the vertex and the two double points, whereas the planes that pass through the double line and one of the lines of kind III intersect the monoid in addition in a transversal. There are four ordinary lines on the monoid in addition to the three lines xy, xz, and yz. There are therefore, in general, six transversals on the monoid; there will be another when three of the ordinary lines or two of these and a line of kind III lie in one plane.

b). If the monoid has a simple line of kind I(1, 2), say yz, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + z^2v_2 + yzsv_1 + z^2sw_1 + xy^2s = 0.$$

The monoid has six ordinary lines on it and will have a transversal whenever three of the nine lines of the monoid lie on one plane.

c). If the monoid has two simple lines of kind I(1, 2), the equation of the monoid will be of the form

$$u_3 + u_2v_1s = 0;$$

where u_3, u_2 and v_1 are homogeneous functions of x, y and z of degrees equal to their subscripts. This monoid does not in general have any transversals.

d). If the monoid has three simple lines of kind I(1, 2,) the equation of the monoid will be of the form

$$u_3 + u_1v_1w_1s = 0;$$

where u_3, u_1, v_1 and w_1 are homogeneous functions of x, y and z of degrees equal to their subscripts. Two of the three planes of the inferior cone each cuts out a line of kind III twice and two lines of kind I, whereas the third plane cuts out two lines of kind I and two ordinary lines, thus making up the twelve lines of the monoid. There are no transversals.

e). If the monoid has a simple line of kind I(1,2), say yz, and a double line of kind II, say $(ax+by, cx+dz)$, the equation of the monoid will be of the form

$$z(ax+by)(exy+fxz+gyz)+(cx+dz)xy+$$
$$s[z(ax+by)^2+y(cx+dz)^2-(ax+by)(cx+dz)(cy+az)]=0.$$

A plane through the two lines of kind III meets the monoid in addition in a conic which passes through the vertex and the two double points; a plane through the double line and either of the two lines of kind III meets the monoid in addition in a transversal which passes through the double point on the line of kind III; and a plane through the double line and either of the two ordinary lines meets the monoid in an additional transversal. If, however, we take the monoid to have for its equation

$$(exy+fxz+gyz)(ax+by)(cx+dz)+$$
$$s[(ax+by)z^2+(cx-dz)yz]=0,$$

where xy and $(ax+by)(cx+dz)$ are the lines of kind III, xz the line of kind II, and yz the line of kind I, the superior cone breaks up into a quadric cone and two planes. In this case the monoid has an additional transversal, for the plane through the two lines of kind III meets the monoid in addition in an ordinary line through the vertex and a transversal.

f). If the monoid has a double line of kind I(2,3), say the line yz, the equation of the monoid will be of the form

$$x^2y^2+u_2z^2+xyzu_1+y^2zs+yz^2s=0.$$

The inferior cone breaks up into three planes that pass through one line. The superior cone has three double edges, of which two are ordinary edges and the third a triple edge on the inferior cone. The plane y cuts the line yz twice out of the monoid and touches the monoid along the line xy, the plane z cuts the line yz twice out of the monoid and touches the monoid along the line xz, and the third plane of the inferior cone cuts out the line yz twice and two ordinary lines. There are in general no transversals on the monoid.

Monoids having three double points on lines of kind III.

a). If this monoid has a double line of kind II, the superior cone has four double lines and must break up. If we call the plane x the single component of the superior cone, xy, xz, and xu_1 (where u_1 is a linear homogeneous function of y and z), the lines of kind III, and yz the double line of kind II, the equation will be of the form

$$x^2u_2+xyzu_1+xv_2s+yzu_1s=0;$$

where u_2 and v_2 are homogeneous quadratic functions of y and z. The plane x meets the monoid in the lines xy, xz, xu_1 and a transversal

xs; the three planes y, z and u_1 each intersect the monoid in the line yz twice, a line of kind III, and a transversal. A plane through the double line and either of the two ordinary lines also meets the monoid in a transversal; the monoid thus has six transversals.

b). If the monoid has a line of kind I(1, 2), the superior cone has three double edges which are simple edges on the inferior cone and a simple edge which is a double edge on the inferior cone. The monoid has four ordinary lines on it and has in general no transversals.

c). If the monoid has two lines of kind I(1, 2), the inferior cone must break up into a quadric cone and a plane. The equation of the monoid will then be of the form

$$u_4 + u_2 u_1 s = 0;$$

where u_4, u_2 and u_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. The single plane of the inferior cone meets the monoid in four lines, of which two are the lines of kind I and two are the two ordinary lines necessary to make up the twelve. This monoid has no transversals.

d). If the monoid has three lines of kind I(1, 2), the inferior cone breaks up into three planes not having a line in common. The equation of the monoid will then be of the form

$$u_4 + u_1 v_1 w_1 s = 0;$$

where u_4, u_1, v_1 and w_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. Each plane of the inferior cone intersects the monoid in a line of kind III and two lines of kind I. There are no transversals.

e). If the monoid has a line of kind I(1, 3) the equation is of the form

$$u_4 + u_3 z + u_2 z^2 + u_1 z^3 + z^4 + v_1 w_1 t_1 s = 0;$$

where v_1, w_1 and t_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. Each of the three planes of the inferior cone meets the monoid in the line of kind I, the line of kind III counting twice and an ordinary line on the monoid. There are no transversals.

f). If the monoid has a double line of kind I(2, 3), the equation of the monoid will be of the form

$$u_1 v_3 + v_1 w_1 t_1 s = 0;$$

where u_1, v_1, w_1, v_3 and t_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. There are of course no other lines on the monoid passing through the vertex; there is however one transversal, and it passes through all three double points.

Monoids having four lines of kind III.

This monoid cannot have a double line of kind II, for this would necessitate the superior cone to break up into components of which two at least are planes. One of these planes would meet the inferior cone in a double line and two single lines, that is, four lines in all, and would therefore form part of it. The monoid would therefore break up.

a). If the monoid has a line of kind I(1, 2), the superior cone breaks up into a plane and a cubic cone that has a double edge. The intersections of the cubic component of the superior cone and the plane and the double edge on the cubic component are single edges on the inferior cone; and an ordinary edge of the cubic component of the superior cone is a double edge on the inferior cone. The plane of the superior cone cuts out of the monoid the three lines of kind III and a transversal which passes through the three double points on these lines.

Monoids having a double point, say (0, 0, 1, 0), *on a line of kind* IV(3, 1).

a). If it has a double line, say xz, of kind II, its equation will be of the form

$$x^2u_2+xzv_2+x^2u_1s+xzv_1s+z^2w_1s=0.$$

The superior cone breaks up into a cubic cone that has a double edge and a plane that passes through this double edge. The inferior cone has a double edge at the double edge of the superior cone and an ordinary edge at the triple edge of the superior cone. The plane of the superior cone cuts out of the monoid the double line, an ordinary line, and a transversal. There are five ordinary lines on the monoid, and no two of them lie in general in the same plane with the double line; there are therefore, in general, five transversals, and they all cross the double line.

b). If the monoid has a second double line of kind II, say the line yz, the equation of the monoid will be of the form

$$x^2y^2+xyzu_1+xyzs+z^2sv_1=0.$$

The superior cone breaks up into a quadric cone and two planes passing through the same edge of the quadric cone and the inferior cone into a plane and a quadric cone. Each of the two planes of the superior cone cuts out the line xy, a double line, and a transversal. There is one ordinary line on the monoid and therefore two more transversals, each crossing this ordinary line and one of the two double lines.

c). If the monoid has a line of kind I(1, 2), say the line xz, the equation of the monoid will be of the form

$$xu_3 + zv_3 + x^2su_1 + xzsv_1 + z^2sw_1 = 0.$$

There are seven ordinary lines on this monoid and in general no transversals.

d). If the monoid has two lines of kind I(1, 2), say xz and yz, the equation of the monoid will be of the form

$$xyu_2 + zu_3 + z^2su_1 + xyzs = 0.$$

The inferior cone breaks up into a quadric cone and a plane which meets the monoid in four lines passing through the vertex. There are five ordinary lines on this monoid and in general no transversals.

The monoid cannot have three lines of kind I(1, 2), for the inferior cone would then break up into three planes, of which one would have to meet the superior cone in the triple edge and two single edges, and would therefore be a component of it.

e). If the monoid has a line of kind I(1, 2), say yz, in addition to a double line of kind II, say xz, the equation will be of the form

$$x^2yu_1 + xzu_2 + xyzs + z^2v_1s = 0.$$

Both cones of the monoid break up into a plane and a cone of order one less. The plane of the superior cone intersects the monoid in the double line, an ordinary line, and a transversal. There are two additional ordinary lines on the monoid, and a plane through either of them and the double line also cuts a transversal from the monoid.

f). If the monoid has a line of kind I(1, 3), say xz, the equation will be of the form

$$xu_3 + zv_3 + x^3s + x^2zs + xz^2s = 0.$$

The inferior cone breaks up into three planes which have a simple edge of the superior cone in common, and one of these three planes also passes through the triple edge of the superior cone. The plane x cuts out the line xy thrice and the line xz once, and the other planes of the inferior cone each cuts out of the monoid three ordinary lines in addition to the line xz. The monoid does not in general have any transversals.

g). If the monoid has a line of kind I(1, 3), say xz, and a line of kind I(1, 2), say the line yz, the equation of the monoid will be of the form

$$xyu_2 + zv_3 + xz^2s = 0.$$

The inferior cone breaks up into a double plane and a single plane. The double plane meets the monoid in the lines xy, xz, and two other lines, which are also lines of kind I(1, 2). Therefore, if the monoid has one line of kind I(1, 2) in addition to the line of kind I(1, 3), it also has two other such lines. This monoid has no transversals.

h). If the monoid has a double line of kind II, say the line xz, and a line of kind I(1, 3), say the line yz, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + yz^2s = 0.$$

The superior cone breaks up into a cubic cone having a double edge and a plane passing through this double edge and the inferior cone into a double plane and a single plane. The single plane which is a component of the superior cone intersects the monoid in the line xy counted once, the line xz counted twice, and a transversal. The double plane of the inferior cone meets the monoid in the double line xz counted twice, the line yz counted once, and another line which is a line of kind I(1, 2). The monoid has four lines and one transversal.

Monoids having a double point, say (0, 0, 1, 0), *on a line of kind* IV(3, 1), *and a double point, say* (0, 1, 0, 0), *on a line of kind* III (2, 1).

a). If the monoid has a double line of kind II, say the line yz, the equation of the monoid will be of the form

$$x^2y^2 + xyzu_1 + xy^2s + z^2sv_1 + yzsw_1 = 0.$$

The plane x intersects the monoid in the line xy, the line xz, an ordinary line through the vertex and a transversal; the plane y cuts out the line xy once, the line yz twice, and a transversal; and the plane z intersects the monoid in the line xz once, the line yz twice, and a transversal. There are two additional ordinary lines on the monoid; a plane through the double line and either of these cuts a transversal from the monoid. The monoid has six lines and three transversals.

b). If the monoid has a double line of kind II and a single line of kind I(1, 2), the equation of the monoid will be of the form

$$u_1v_1w_2 + u_2w_1s = 0;$$

where u_1, v_1, w_1, u_2 and w_2 are homogeneous functions of x, y and z of degrees equal to the subscripts. One of the two planes of the superior cone intersects the monoid in the line of kind IV and in the double line, and therefore intersects the monoid also in a transversal; the other plane of the superior cone intersects the monoid in addition to a transversal in the line of kind IV, the single line of kind I, and an ordinary line which is the line necessary to make up the twelve lines on the monoid.

c). If the monoid has a line of kind I(1, 3), say the line yz, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + y^2zs + yz^2s = 0.$$

The plane x, which with a cubic cone constitutes the superior cone of the monoid, intersects the monoid in three lines through the vertex and in a transversal. The three planes into which the inferior cone breaks up each meets the monoid in four lines, and therefore touches the monoid at the vertex. The monoid has in general only one transversal. The lines of the monoid of course lie in the three planes of the inferior cone.

d). If the monoid has a line of kind I(1,2), say the line yz, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + xy^2s + yzsv_1 + z^2sw_1 = 0.$$

There are five ordinary lines on the monoid, one of which lies in the same plane as the lines xy and xz; this plane cuts out a transversal. There is in general no other transversal.

e). If the monoid has two lines of kind I(1,2), the equation of the monoid will be of the form

$$u_1v_3 + u_2v_1s = 0;$$

where u_1, u_2, v_1 and v_3 are homogeneous functions of x, y and z of degrees equal to their subscripts. The double and triple edges of the superior cone are ordinary edges on the inferior cone, and the double edges on the inferior cone are ordinary edges on the superior cone. The plane of the inferior cone meets the monoid in the two lines of kind I and two ordinary lines. The remaining ordinary line necessary to make up the twelve lines of the monoid is the remaining intersection of the quadric component of the inferior cone and the cubic component of the superior cone. The plane of the superior cone meets the monoid in a transversal; there is in general no other transversal.

Monoids having a line of kind IV(3,1) *and two lines of kind* III(2,1).

This monoid cannot have a double edge of kind II, for the superior cone that has a triple edge and three double edges breaks up into three planes through a line and a fourth plane not through this line; the inferior cone having one double edge of the superior cone as a double edge and the two other double edges as simple edges, must contain one of the planes of the superior cone as factor. This causes the monoid to break up.

a). If the monoid has a line of kind I(1,2), the superior cone must break up into a quadric cone and two planes that have an edge of the quadric cone in common. The double edge of the inferior cone is an ordinary edge on the quadric cone and the triple edge on the superior cone is an ordinary edge on the quadric cone. These four

lines count for nine lines of the monoid. Each plane of the superior cone intersects the monoid in the line of kind IV, a line of kind III, an ordinary line on the monoid, and a transversal. The monoid has two transversals, each passing through the double point on the line of kind IV and one of the double points on the lines of kind III. The twelfth line on the monoid is the residual intersection of the quadric cone and the cubic cone. The equation of the monoid will be of the form

$$u_1v_1w_2 + u_3s = 0;$$

where u_1, u_3, v_1 and w_2 are homogeneous functions of x, y and z of degrees equal to their subscripts.

Monoids having two lines of kind IV(3, 1), *say* xy *and* xz.

a). If the monoid has a double line of kind II, say yz, the equation of the monoid will be of the form

$$x^2yz + xy^2s + yzsu_1 + z^2sv_1 = 0.$$

The superior cone breaks up into a double plane and two single planes. The inferior cone has the two triple edges of the superior cone as single edges and the double edge of the superior cone as a double edge. The double plane of the superior cone meets the monoid in the two lines of kind IV, a transversal and a line which, being a double line on the superior cone and a single line on the inferior cone, is a line of kind III on the monoid. The single planes of the superior cone intersect the monoid in the double line counted twice, a line of kind IV, and a transversal. This monoid thus has four lines on it and four transversals, for a plane through the double line and the line of kind III also cuts a transversal out of a monoid.

b). If the monoid has a line of kind I(1, 2), say the line yz, the the equation of the monoid will be of the form

$$x^3y + x^2zu_1 + xy^2s + yzsv_1 + z^2sw_1 = 0.$$

The superior cone breaks up into a double plane and a quadric cone. The two triple edges on the superior cone are ordinary edges on the inferior cone, and the double edge on the inferior cone is a single edge on the superior cone. The double plane meets the inferior cone in the two lines of kind IV and a line of kind III. The plane x thus cuts out of the monoid three lines to double points and a transversal passing through these double points. There is in general no other transversal. The remaining two lines necessary to make up the twelve lines of the monoid are the residual intersections of the quadric and the cubic cones.

c). If the monoid have a line of kind I(1, 3), say the line yz, the equation of the monoid will be of the form

$$x^3y + x^2zu_1 + y^2zs + yz^2s = 0.$$

The superior cone breaks up into a double plane and a quadric cone and the inferior cone into three planes that have an edge of the quadric cone in common. The double plane of the superior cone intersects the monoid in the two lines of kind IV and a line of kind III, which is therefore a line to a double point. As in the previous cases, there is one transversal, and it passes through all three double points. Two of the single planes of the inferior cone intersect the monoid in the line of kind IV counted thrice and the line of kind I, whereas the third plane intersects the monoid in the line of kind I, the line of kind III counted twice, and an ordinary line.

d). If the monoid has two lines of kind I(1,2), the equation will be of the form

$$u_1^2v_2 + u_2w_1s = 0;$$

where u_1, u_2, v_2 and w_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. The triple lines on the superior cone are ordinary lines on the inferior cone and the double lines on the inferior cone are ordinary lines on the superior cone. The double plane of the superior cone intersects the monoid in the two lines of kind IV and a line of kind III; the monoid thus has another double point. As in previous cases, there is a transversal passing through these three double points. The monoid thus has five lines on it and one transversal.

Monoids having a line of kind IV(4,1), *say the line* xy.

a). If the monoid has a double line of kind II, say the line xz, the equation of the monoid will be of the form

$$x^2u_2 + x^2su_1 + xzsv_1 + z^2sw_1 = 0.$$

The superior cone breaks up into a double plane and two single planes, all having the line xy in common. The quadruple line on the superior cone is an ordinary line on the inferior cone, and a double edge on the superior cone is a double edge on the inferior cone. The double plane of the superior cone intersects the monoid in the line xy, the line xz counted twice, and a transversal. Each of the two single planes of the superior cone intersects the monoid in the line xy, two ordinary lines and a transversal. A plane through the double line and each of these four ordinary lines of the monoid also cuts a transversal out of the monoid. The monoid thus has six lines and seven transversals.

b). If the monoid has a second double line of kind II, say the line yz, the equation of the monoid will be of the form

$$x^2y^2 + xyzs + z^2su_1 = 0.$$

The superior cone breaks up into two double planes and the inferior cone into a quadric cone and a plane. The inferior cone has the

quadruple line of the superior cone as an ordinary edge and the double lines as double edges, Each of the double planes of the superior cone cuts out of the monoid a double line, the line of kind IV and a transversal. The monoid has three lines and two transversals.

c). If the monoid has a line of kind I(1, 2), say the line xz, the equation of the monoid will be of the form

$$xu_3 + x^2u_1s + xzsv_1 + w_1z^2s = 0.$$

The plane x intersects the monoid in the line xy, the line xz counted twice, and a transversal. Each of the other three planes of the superior cone intersects the monoid in the line of kind IV and two ordinary lines (thus making up the twelve lines on the monoid), and also a transversal. There are thus eight lines on the monoid, and in general only four transversals, and each of these four passes through the double point on the line of kind IV.

d). If the monoid has a second line of kind I(1, 2), say the line yz, its equation will be of the form

$$xyu_2 + z^2u_1s + xyzs = 0.$$

The plane x intersects the monoid in the line xy, the line xz counted twice, and a transversal; the plane y intersects the monoid in the line xy, the line yz counted twice, and a transversal; and each of the other two planes of the superior cone intersects the monoid in the line xy and two ordinary lines through the vertex in addition to the transversal. The monoid thus has seven lines and in general only four transversals.

e). If this monoid has a double line of kind II, say the line xz, and a line of kind I(1, 2), say the line yz, its equation will be of the form

$$x^2yu_1 + xyzs + v_1z^2s = 0.$$

The double plane of the superior cone intersects the monoid in the line of kind IV, the line of kind II counted twice, and a transversal; each of the two single planes cut out the line of kind IV, the line of kind I counted twice, and a transversal; and the other single plane intersects the monoid in the line of kind IV, two ordinary lines, and a transversal. A plane through the double line and the ordinary line which does not lie in the plane of the inferior cone also cuts a transversal from the monoid. This monoid has five lines on it and four transversals.

Monoids having a line of kind IV(4, 1), *say* xy, *and a line of kind* III(2, 1), *say* xz.

a). If the monoid has a double line of kind II, say the line yz, its equation will be of the form

$$x^2y^2 + u_1z^2s + v_1yzs + xy^2s = 0.$$

The superior cone breaks up into two double planes. One of these intersects the monoid in the line xy, the line yz counted twice, and a transversal, and the other cuts out the lines xy and xz and a transversal, in addition to a line which is a second line of kind III. A plane through the double line and either of the lines of kind III will cut out an additional transversal that passes through the double point on the line of kind III. This monoid has four lines and four transversals.

b). If the monoid has a line of kind I(1,2), say the line yz, its equation will be of the form

$$x^2yu_1 + xy^2s + v_1yzs + w_1z^2s = 0.$$

The superior cone breaks up into a double plane and two single planes. The double plane intersects the monoid in the lines xy and xz and a transversal in addition to a line which, being double on the superior cone and single on the inferior cone, is a line of kind III. Each of the other two planes of the superior cone intersects the monoid in three lines through the vertex and a transversal. There are thus six lines on the monoid and in general only three transversals.

Monoids having a double line of kind III(2,3), *say the line* xy.

a). If the monoid has a line of kind I(1,2), say the line xz, its equation will be of the form

$$xu_3 + zv_3 + u_1x^2s + v_1xzs = 0.$$

The plane x intersects the monoid in the line xy thrice and the line xz. Each plane through the line xy and one of the four ordinary lines of the monoid cuts out a transversal. The monoid thus has six lines on it and in general just four transversals, and these four all pass through the triple point on the line of kind III.

b). If the monoid has a line of kind I(1,3), say the line xz, its equation will be of the form

$$xu_3 + zv_3 + x^3s + x^2zs = 0.$$

The inferior cone breaks up into a double plane and a single plane. The single plane intersects the monoid in three ordinary lines in addition to the line of kind I. A plane through the double line and either the line of kind I or one of the ordinary lines cuts a transversal from the monoid. This monoid thus has six lines and four transversals passing through the triple point on the line of kind III.

Monoids having a double line of kind III, *say the line* xy, *and a line of kind* III(2,1), *say the line* xz.

a). If the monoid has a line of kind I(1,2), say the line yz, the equation of the monoid will be of the form

$$x^2yu_1 + xzu_2 + yzsv_1 + xy^2s = 0.$$

The plane x intersects the monoid in the line xy counted twice, the line xz, and a transversal. Each plane through the double line and one of the two ordinary lines on the monoid intersects the monoid in addition in a transversal. The monoid thus has five lines and three transversals.

b). If the monoid has a line of kind I(1, 3), say the line yz, its equation will be of the form

$$x^2yu_1 + xzu_2 + y^2zs = 0.$$

The plane of the inferior cone intersects the monoid in the line yz, the line xz counted twice, and an ordinary line of the monoid. The plane of the superior cone cuts out of the monoid the line xy twice, the line xz, and a transversal. The double plane of the inferior cone cuts out the line xy thrice and the line yz once. A plane through the double line and the ordinary line also cuts out a transversal. This monoid has four lines and two transversals.

Monoids having a triple point on a line of kind III *and two double points on lines of kind* III.

a). If the monoid has a line of kind I(1, 2), its equation will be of the form

$$u_2u_1v_1 + v_2w_1s = 0;$$

where u_2, u_1, v_2, v_1 and w_1 are homogeneous functions of x, y and z of degrees equal to the subscripts. Here u_1 and v_1 have an edge of u_2 in common. The triple edge of the superior cone is a double edge on the inferior cone, the two double edges of the superior cone are single edges on the inferior cone, and the second double edge of the inferior cone is a simple edge on the superior cone. There are no other lines on the monoid. Each of the two single planes of the superior cone intersects the monoid in a transversal.

Monoids having a double line of kind III, *say the line* xy, *and a double point on a line of kind* IV(3, 1).

a). If the monoid has a line of kind I(1, 2), say the line yz, its equation will be of the form

$$x^3y + x^2zu_1 + xy^2s + yzsv_1 = 0.$$

The superior cone breaks up into a double plane and a quadric cone and the inferior cone into a plane and a quadric cone. There is one ordinary line on the monoid. The double plane cuts out the line of kind IV, the double line, and a transversal. A plane through the double line and the ordinary line also cuts out a transversal. The monoid thus has four lines and two transversals.

b). If the monoid has a line of kind I(1,3), say the line yz, its equation will be of the form

$$x^3y + x^2zu_1 + y^2zs = 0.$$

The superior cone breaks up into a double plane and a quadric cone; the inferior cone breaks up into a double plane and a single plane. The double plane of the superior cone intersects the monoid in the double line, the line of kind IV, and a transversal. There are only three lines on the monoid and one transversal.

Monoids having a line of kind IV(4,1), *say the line* xy, *and a line of kind* IV(3,1), *say the line* xz.

a). If the monoid has a line of kind I(1,2), say the line yz, the equation will be of the form

$$x^3y + xy^2s + v_1yzs + w_1z^2s = 0.$$

The triple plane of the superior cone intersects the monoid in the line xy, the line xz, and a third line which, being a triple line on the superior cone and an ordinary line on the inferior cone, is also a line of kind IV; this plane also cuts out a transversal. The single plane of the superior cone also cuts a transversal from the monoid. The monoid thus has four lines and two transversals.